SQ28 - Spring 2017

Partial credit will be assigned based upon the correctness, completeness, and clarity of your answers. ***Correct answers without proper justification will not receive any credit.***

Exercise 1 (Benoît and the mosquito)**.** Benoît has a hard time falling asleep when there is a mosquito in his bedroom as it will fly near his ears from time to time. Denote by X a random variable that represents the time Benoît needs to fall asleep and by Y the time before a mosquito attack. Assume that:

$$X \sim \mathcal{E}(\lambda) \quad \text{and} \quad Y \sim \mathcal{E}(\mu),$$

where $\lambda, \mu \in \mathbb{R}_+$. Assume also that X and Y are independent.

1. Show that the exponential distribution is memoryless, i.e. for all $s, t > 0$:
$$\mathbb{P}_{(X>s)}(X > s + t) = \mathbb{P}(X > t)$$
2. Does the model seem legitimate?
3. One can show that:
$$\mathbb{P}(X < Y) = \int_0^{+\infty} \mathbb{P}(X < y)\mu e^{-\mu y}\, dy.$$
   (a) Compute $\mathbb{P}(X < y)$ for all $y \in \mathbb{R}_+$.

   (b) Deduce that:
$$\mathbb{P}(X < Y) = \frac{\lambda}{\lambda + \mu}$$
   (c) Give a similar result for $\mathbb{P}(Y < X)$.

   (d) Deduce $\mathbb{P}(X = Y)$.
4. From now on, we will assume that there are 20 mosquitos in Benoît's bedroom and that the times of the attacks are still exponentially distributed with parameter μ. We will also assume that the attacks are mutually independent and that Benoît has ninja skills: he kills each mosquito after its first attack. The aim of the following questions is to give a confidence interval for μ. Denote by $(X_1, \ldots, X_{20})$ the times of the attacks.

   (a) One can show that if $(X_1, \ldots, X_n)$ is a random sample of $\mathcal{E}(\mu)$ then:
$$\mu \times \left(\sum_{i=1}^{n} X_i\right) \sim \mathrm{Er}(n),$$
   where Er denotes the Erlang distribution. Find α and β in the Erlang table[1] such that:
$$\mathbb{P}(\alpha \leqslant \mathrm{Er}(20) \leqslant \beta) = 95\%.$$
   (b) Deduce a confidence interval at a level of 95% for μ.

   (c) As Benoît couldn't sleep last night with his 20 mosquitos, he took the time to write down the times of the attacks. The empirical mean he observed was $\bar{x} = 5.2$ minutes. Give the corresponding observed confidence interval.

Exercise 2 (Scilab)**.** In this exercise, you will write a program to estimate the probability $\mathbb{P}(X < Y)$ from exercise 1. This exercise is independent from exercise 1. If you don't remember the synthax of a command, just write it in plain english.

1. Write a Scilab program that takes two parameters λ and μ and simulates X and Y where
$$X \sim \mathcal{E}(\lambda) \quad \text{and} \quad Y \sim \mathcal{E}(\mu)$$
   Your program will output 1 if $X < Y$ and 0 otherwise.
2. Write a program that gives an estimation of $\mathbb{P}(X < Y)$: on what result from the course is this estimation based?

[1] the Erlang table works exactly like the one of the χ^2 distribution, you will find it on page 2.

Exercise 3. Damien has a deck of 32 cards. He decides to draw 5 cards at once (randomly).

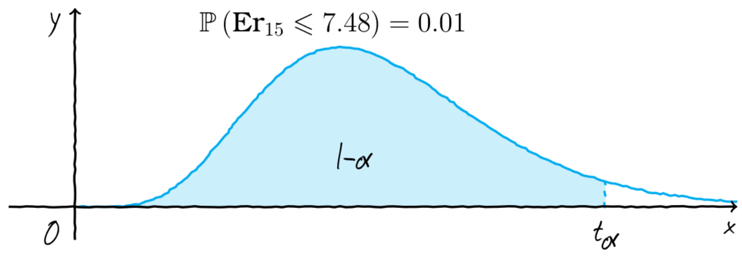
1. What is the probability that he will get exactly 2 queens and one king?
2. What is the probability that he will get at least one king?

Exercise 4. Let $a \in \mathbb{R}$ and let f be the function:

$$f : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto \begin{cases} a & \text{if } 0 \leqslant x \leqslant 1, \\ 2a & \text{if } 1 < x \leqslant 2, \\ 0 & \text{otherwise.} \end{cases}$$

1. Plot f.
2. Find a such that f is a probability density function.
3. Let X be a continuous random variable with pdf f.
   (a) Compute $\mathbb{P}(X \leqslant 1)$.
   (b) Compute $\mathbb{E}(X)$.
   (c) Compute the cumulative distribution function of X.

Erlang table



k	0.005	0.01	0.025	0.05	0.1	0.9	0.95	0.975	0.99	0.995
1	0.01	0.01	0.03	0.05	0.11	2.3	3.0	3.69	4.61	5.3
2	0.1	0.15	0.24	0.36	0.53	3.89	4.74	5.57	6.64	7.43
3	0.34	0.44	0.62	0.82	1.1	5.32	6.3	7.22	8.41	9.27
4	0.67	0.82	1.09	1.37	1.74	6.68	7.75	8.77	10.05	10.98
5	1.08	1.28	1.62	1.97	2.43	7.99	9.15	10.24	11.6	12.59
6	1.54	1.79	2.2	2.61	3.15	9.27	10.51	11.67	13.11	14.15
7	2.04	2.33	2.81	3.29	3.89	10.53	11.84	13.06	14.57	15.66
8	2.57	2.91	3.45	3.98	4.66	11.77	13.15	14.42	16.0	17.13
9	3.13	3.51	4.12	4.7	5.43	12.99	14.43	15.76	17.4	18.58
10	3.72	4.13	4.8	5.43	6.22	14.21	15.71	17.08	18.78	20.0
11	4.32	4.77	5.49	6.17	7.02	15.41	16.96	18.39	20.14	21.4
12	4.94	5.43	6.2	6.92	7.83	16.6	18.21	19.68	21.49	22.78
13	5.58	6.1	6.92	7.69	8.65	17.78	19.44	20.96	22.82	24.14
14	6.23	6.78	7.65	8.46	9.47	18.96	20.67	22.23	24.14	25.5
15	6.89	7.48	8.4	9.25	10.3	20.13	21.89	23.49	25.45	26.84
16	7.57	8.18	9.15	10.04	11.14	21.29	23.1	24.74	26.74	28.16
17	8.25	8.89	9.9	10.83	11.98	22.45	24.3	25.98	28.03	29.48
18	8.94	9.62	10.67	11.63	12.82	23.61	25.5	27.22	29.31	30.79
19	9.64	10.35	11.44	12.44	13.67	24.76	26.69	28.45	30.58	32.09
20	10.35	11.08	12.22	13.25	14.53	25.9	27.88	29.67	31.85	33.38
21	11.07	11.83	13.0	14.07	15.38	27.05	29.06	30.89	33.1	34.67
22	11.79	12.57	13.79	14.89	16.24	28.18	30.24	32.1	34.35	35.95
23	12.52	13.33	14.58	15.72	17.11	29.32	31.41	33.31	35.6	37.22
24	13.26	14.09	15.38	16.55	17.97	30.45	32.59	34.51	36.84	38.48
25	14.0	14.85	16.18	17.38	18.84	31.58	33.75	35.71	38.08	39.74
26	14.74	15.62	16.98	18.22	19.72	32.71	34.92	36.9	39.31	41.0
27	15.49	16.4	17.79	19.06	20.59	33.84	36.08	38.1	40.53	42.25
28	16.25	17.17	18.61	19.9	21.47	34.96	37.23	39.28	41.76	43.5
29	17.0	17.96	19.42	20.75	22.35	36.08	38.39	40.47	42.98	44.74
30	17.77	18.74	20.24	21.59	23.23	37.2	39.54	41.65	44.19	45.98
40	25.59	26.77	28.58	30.2	32.14	48.29	50.94	53.31	56.16	58.16
50	33.66	35.03	37.11	38.96	41.18	59.25	62.17	64.78	67.9	70.08
60	41.93	43.46	45.79	47.85	50.31	70.12	73.28	76.11	79.48	81.82
70	50.33	52.02	54.57	56.83	59.51	80.91	84.31	87.32	90.92	93.42
80	58.84	60.67	63.44	65.88	68.77	91.66	95.26	98.46	102.27	104.91
90	67.44	69.41	72.37	74.98	78.08	102.35	106.15	109.52	113.53	116.31
100	76.12	78.22	81.36	84.14	87.42	113.01	117.0	120.53	124.72	127.63