Probability and Statistics - SQ28
Tronc commun
Midterm - Spring 2014

Test duration : 2 hours

Partial credit will be assigned based upon the correctness, completeness, and clarity of your answers. Correct answers without proper justification will not receive full credit.

An A4 sheet with handwritten notes is allowed, as is the table of the standard normal distribution.
Calculators are allowed.

Exercise 1 (6 points)

The three following questions are independent.

1. A jury is made of 6 randomly chosen persons in a group made of 6 men and 4 women. We denote by X the random variable counting the number of women in the jury. What is the probability distribution of X? Compute $\mathbb{E}(X)$.

2. One rolls three fair dice and denotes by Y the number of different faces (*e.g.* if the result of the roll is 3, 4 and 2 then $Y = 3$, if it is 4, 4 and 6 then $Y = 2$, etc...).
   What is the probability distribution of Y? Compute its expected value.
   The results should be given as irreductible fractions.

3. Let Z be a continuous random variable with probability density function given by:
$$f(t) = \begin{cases} \dfrac{2}{t^3} & \text{if } t \geqslant 1, \\ 0 & \text{if } t < 1. \end{cases}$$
   Compute the cumulative distribution function of Z. Deduce the median of Z, i.e. the real number m such that $\mathbb{P}(Z \leqslant m) = \mathbb{P}(Z > m)$.

Exercise 2 (14 points)

A table tennis ball factory uses two machines, A and B. Machine A generates a third of the production, while machine B generates the rest. 12% of the balls produced by machine A are defective as are 9% of the ones coming from machine B. At the end of the production line, the balls are mixed up so that if one would take a ball randomly, it would have a probability 1/3 of having been made by machine A.

1. One takes a random ball at the end of the production line. Denote by:

   - A : «the ball came from machine A»,
   - B : «the ball came from machine B»,
   - D : «the ball is defective».

   (a) Show that $\mathbb{P}(D) = 1/10$.

   (b) Suppose that the ball is defective. What is the probability that it was produced by machine A?

2. Let $n \in \mathbb{N}^\star$ and suppose that one draws randomly n balls, where the draws are supposed to be mutually independent. Let X denote the total number of defective balls.

   (a) What is the probability distribution of X? Describe $X(\Omega)$ and for each $k \in X(\Omega)$, give $\mathbb{P}(X = k)$.

   (b) Compute the expected value of X and its variance $\mathbb{V}\text{ar}(X)$.

3. In this question, we assume that $n = 3600$. We admit that X can be approximated using a normal random variable $Z \sim \mathcal{N}(\mu, \sigma^2)$.

   (a) Find μ and σ^2 such that Z and X have the same expected value and the same variance.

   (b) Give an approximation of the probability that at least 350 balls are defective.

4. We stop the production from machine B and suppose that the number of balls produced by machine A in five minutes is a random variable Y Poisson-distributed with parameter $\lambda = 20$.

   We denote by T the number of defective balls made by machine A in 5 minutes.

   (a) What values can Y take? For each $n \in Y(\Omega)$, give $\mathbb{P}(Y = n)$. Give also $\mathbb{E}(Y)$ and $\mathbb{V}\text{ar}(Y)$.

   (b) What is the average number of balls produced by machine A in an hour?

   (c) Let k and n be two integers. By distinguishing whether $k \leqslant n$ or $k > n$, compute the probability $\mathbb{P}(T = k | Y = n)$.

   (d) Using the partition $(Y = n)_{n \in \mathbb{N}}$, show that T is Poisson-distributed with parameter $\gamma = 2.4$.