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Probability and Statistics - SQ28
Tronc commun
Midterm- Spring 2015

Test duration : 2 hours

Partial credit will be assigned based upon the correctness, completeness, and clarity of your answers. Correct answers without proper justification will not receive full credit.

Calculators are not allowed.

Exercise 1 4 points

1. Let A and B be two events. Recall the definition of the above assertions:
   (a) A and B are independent.
   (b) A implies B.
2. Assume that A and B satisfy:
$$\mathbb{P}(A) = \frac{1}{5} \qquad \text{and} \qquad \mathbb{P}(B) = \frac{2}{3}.$$
   Compute $\mathbb{P}(A \cup B)$ in each of the following situations:
   (a) A and B are exclusives.
   (b) A and B are independent.
   (c) A implies B.
   (d) $P_B(A) = \frac{1}{2}$.

Exercise 2 6 points

Maxime has a chicken. Every night, it lays one egg with probability 3/4 or doesn't lay any egg with probability 1/4, independently from one night to another.

1. Let X be the number of eggs after 7 nights.
   (a) What is the probability distribution of X?
   (b) Maxime decides that he will sell all his eggs next week. He sells an egg at \$0.2. Denote by Y his winnings.
      i. What is the link between X and Y?
      ii. How much can Maxime expect to win next week?

One year has passed and Maxime's chicken looks bad: bird flu got to it. It will now lay an egg with probability 1/20.

(a) Denote by Z the number of nights Maxime has to wait before his chicken lays its first egg. What is the probability distribution of Z?

(b) Marcelle will be visiting Maxime in 7 days. Compute the probability that Maxime will have at least 1 egg to make her crepes.

Exercise 3 10 points

The number of vehicles arriving to a toll[1] between time 0 and time t is denoted by N_t. We suppose that $N_t \sim \mathcal{P}(\lambda t)$ where $\lambda > 0$ is a known parameter.

1. What is the expected value of N_t? What is its variance?

2. Let X_1 be the arrival time of the first vehicle.

   (a) Let $t > 0$. What is the link between the events $(X_1 > t)$ and $(N_t = 0)$? Deduce $\mathbb{P}(X_1 > t)$.

   (b) What is the probability distribution of X_1?

   (c) Give $\mathbb{E}(X_1)$ and $\mathbb{V}\mathrm{ar}(X_1)$.

3. For all $n \in \mathbb{N}^\star$, denote by X_n the arrival time of the n-th vehicle to the toll (after time 0). Denote also by F_n the cumulative distribution function[2] of X_n.

   (a) Compute $F_n(t)$ for all $t \leqslant 0$.

   (b) Let $t > 0$ and $n \in \mathbb{N}^\star$. Explain why:

   $$X_n \leqslant t \quad \Longleftrightarrow \quad N_t \geqslant n.$$

   (c) Deduce $F_n(t)$ for all $t \in \mathbb{R}$.

   (d) Show that a probability density function of X_n is given by:

   $$f_n(t) = \begin{cases} \dfrac{\lambda^n}{(n-1)!}\, t^{n-1}\, \mathrm{e}^{-\lambda t} & \text{if } t > 0, \\ 0 & \text{otherwise.} \end{cases}$$

[1] péage

[2] For all $t \in \mathbb{R}$, $F_n(t) = \mathbb{P}(X_n \leqslant t)$