Sample test

Partial credit will be assigned based upon the correctness, completeness, and clarity of your answers. ***Correct answers without proper justification will not receive any credit.***

Exercise 1. Let A and B be two independent events such that:

$$\mathbb{P}(A) = 0.2, \quad \mathbb{P}(B) = 0.4$$

Compute $P(A \cup B)$.

Exercise 2. You draw 6 cards from a well-shuffled 52-cards deck. Compute the probability of the following events.

1. You draw at least two aces.

2. You draw exactly one king and two queens.

Exercise 3. 10% of Maxime's emails are spams. Amongst those spams, 80% contain the word "pill" whereas only 5% of Maxime's regular (non-spam) emails contain this word. Maxime just received an email containing the word "pill". Compute the probability that this email is a spam.

Exercise 4. Damien has 30 socks in his drawer. Five of them are blue. He picks one at random and if it's not blue, he puts it back until he gets a blue one.

1. What is the average number of trials Damien will need to get his blue sock?

2. After a few unsuccessful attemps, Damien is angry: he decides to draw 10 socks at the same time. What is the probability that he will get at least one blue sock?

Exercise 5. Let X be a random variable with cumulative distribution function:

$$F_X(t) = \begin{cases} 0 & \text{if } t < 0, \\ 2/3 & \text{if } 0 \leqslant t < 1 \\ 1 & \text{otherwise.} \end{cases}$$

Plot F_X. What is the probability distribution of X?

Solution 1. As A and B are independent, $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$. Hence:

$$\begin{aligned}\mathbb{P}(A \cup B) &= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \\ &= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A)\mathbb{P}(B) \\ &= 0.2 + 0.4 - 0.2 \times 0.4\end{aligned}$$

If you don't write somewhere that A and B are independent then you will get partial credit.

Solution 2. 1. The number of hands of 6 cards with exactly two aces is

$$\underbrace{\binom{4}{2}}_{\text{choosing 2 aces}} \times \underbrace{\binom{48}{4}}_{\text{choosing 4 other cards}}$$

It is necessary to explain your answer. If you only write the result then you won't get *any* credit.

Thus, the probability of drawing at least two aces is:

$$\underbrace{\frac{\binom{4}{2}\binom{48}{4}}{\binom{52}{6}}}_{\text{exactly 2 aces}} + \underbrace{\frac{\binom{4}{3}\binom{48}{3}}{\binom{52}{6}}}_{\text{exactly 3 aces}} + \underbrace{\frac{\binom{4}{4}\binom{48}{2}}{\binom{52}{6}}}_{\text{exactly 4 aces}}$$

2. The probability of drawing exactly one king and two queens is:

$$\frac{\binom{4}{1}\binom{4}{2}\binom{44}{3}}{\binom{52}{6}}.$$

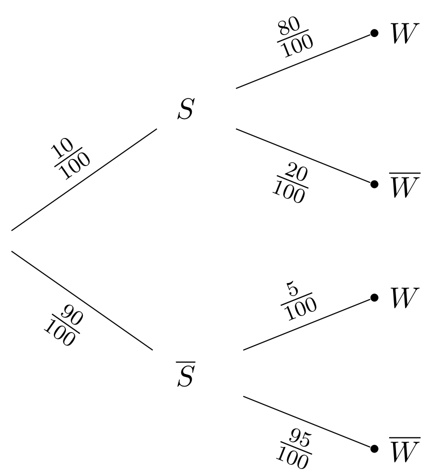
If the reasoning is similar to a question already answered then it is not necessary to explain the result.

Solution 3. Denote by:

- S: "the email is a spam"
- W: "the email contains the word 'pill'"

Always take the time to explain new notations: I shouldn't have to guess what the letters mean.

The corresponding probability tree is:



Thus:

$$\mathbb{P}_W(S) = \frac{\mathbb{P}(S \cap W)}{\mathbb{P}(W)} = \frac{\frac{10}{100} \cdot \frac{80}{100}}{\frac{10}{100} \cdot \frac{80}{100} + \frac{90}{100} \cdot \frac{5}{100}} = \frac{16}{25}$$

Solution 4. 1. Denote by X the number of trials Damien will need to get to his blue sock. We are looking for $\mathbb{E}(X)$. Here, X has a geometric probability distribution with parameter $1/6$ because it counts the number of attempts before the first success in a repetition of independent Bernoulli experiments with probability of success $1/6$. Hence:

$$\mathbb{E}(X) = \frac{1}{1/6} = 6.$$

When you recognize one of the usual distributions, give its names and parameters and explain with a little sentence.

2. Denote by Y the number of blue socks Damien gets. Then, Y is a hypergeometric variable as it's a draw without replacement. The corresponding parameters are 30 (total number of socks), 10 (number of draws) and 5 (number of blue socks):

$$Y \sim \mathcal{H}(30, 10, 5).$$

Thus:

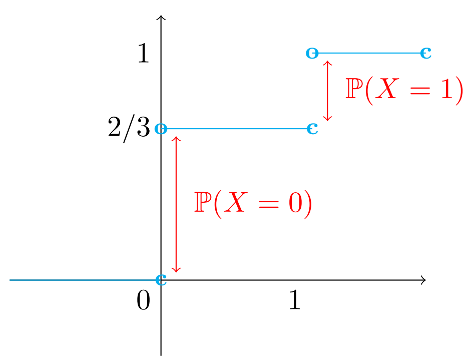
$$\mathbb{P}(Y \geqslant 1) = 1 - \mathbb{P}(Y = 0) = 1 - \frac{\binom{5}{0}\binom{25}{10}}{\binom{30}{10}}.$$

Solution 5. Recall that:

$$\forall t \in \mathbb{R}, \quad F_X(t) = \mathbb{P}(X \leqslant t).$$

Let's plot F_X:

Sometimes it is easier to explain a result with a drawing. Don't hesitate to annotate a graph.



Hence, $X \sim \mathcal{B}(1/3)$.