$\mathbb{P}\left(\sum_{j=1}^{J}\frac{(N_{\hat{p}_j}-N_{p_j})^2}{N_{p_j}} \leqslant \chi^2_{J-1,\alpha}\right) \simeq 1-\alpha$

$\frac{\overline{X}-\mu}{\sigma} \to \mathcal{N}(0,1)$

$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \mathrm{e}^{-x^2/2}\mathrm{d}x = 1$

$\mathbb{E}(\varphi(X)) = \int \varphi(x)\mathrm{d}\mathbb{P}_X(x)$

$\binom{n}{k}p^k(1-p)^{n-k}$

$\mathbb{P}(A)=\sum_{i\in I}\mathbb{P}_{B_i}(A)\mathbb{P}(B_i)$

1 Describe the sample space in the following situations :

1. You toss a coin three times ;
2. You roll two dice ;
3. You pick up randomly two cards from an ordinary 52-card deck (suits are not taken into account).

2 Someone rolls two non-weighted dice.

1. Assume that there is a red die and a green one. Give a sample space Ω_1 corresponding to this experiment.
2. To what subset of Ω_1 corresponds the event «the sum of the two dice is 7» ?
3. Find another sample space Ω_2 to describe the experiment (imagine that the person rolling the dice is color-blind).
4. What subset of Ω_2 corresponds to the event «the sum of the two dice is 7» ? What is the probability of this event in this case ? Is it the same as in the previous model ?

3 There are 5 seats around a table and 4 people to be seated at the table. In how many ways can they seat themselves ?

4 During the 100m world championship, 8 athletes run for the gold. How many different possible podia can there be at the end of the run ?

5 Two cards are drawn from a well-shuffled deck of 52 cards. What is the probability that both are aces if the card is replaced ? What if it's not replaced ?

6 In a deck of 52 cards, how many hands of four cards are there with :

1. exactly one king ;
2. no ace ;
3. at least one king ;

7 I have 100 mp3's on my computer and I decide to copy 10 of them on my USB key to listen to music in my car. How many selections can I make :

a. if I take the order of the tracks into account ?

b. if I don't care about the order of the tracks ?

8 A coin is tossed twice. Let us denote by A,B,C the following events :

- A : heads is obtained in the first toss ;
- B : heads is obtained in the second toss ;
- C : exactly one heads is obtained after two tosses ;

Show that A,B,C are pairwise independent but not independent.

9 Let us consider a rare disease affecting 1% of the population. Assume there exists a test which gives positive reaction to 80% of the people having the disease and to 10% of the people who do not. Given that the test has a positive reaction on someone, what is the probability that the person is sick ?

10 How many words can you make with the letters "ABC" ? With the letters "AABC" ? With the letters "STATISTICS" ?

11 In a batch of 100 dice, 25 are loaded and will land on a 6 with probability 1/2, and 75 are fair. You take a die randomly, roll it and get a 6. What is the probability that this die is crooked ?

12 A shepherd has 100 sheep and decides to shave 3 of them.

1. How many possibilities are there ?
2. What is the probability that, picking randomly (uniformly), one (and only one) of his two oldest sheep will be shaved ?
3. What is the probability that, picking randomly (uniformly), his favorite sheep Randy will be shaved ?

13 How many five letters words are there :

1. if letters can be repeated ?
2. if no letter can be repeated ?
3. that start with the letter `B` if letters can't be repeated ?
4. that start with the letter `B` and ends with an `S` if letters can be repeated ?

14 When rolling two dice, what is the probability of the following events ?

1. The sum of the dice is equal to 7.
2. You get at least one 6.
3. You get exactly one 6.

15 Urn 1 contains 2 red balls and 5 blue balls. Urn 2 contains 5 red balls and 1 blue. You pick an urn randomly (both urns have a 50% chance of being chosen) and draw a blue ball. What is the probability that it came from urn 1 ?

16 Two Face[1] has two coins in his pocket, a regular one and a two-headed one. He picks one at random from his pocket, tosses it and obtains heads. What is the probability that he flipped the fair coin ?

17 There are 9 students in a class : 6 boys and 3 girls. If the teacher picks a group of 3 at random, what is the probability that everyone in the group is a boy ?

18 If you flip a fair coin 7 times, what is the probability that you will get exactly 5 tails ?

19 Let A and B be two events such that $\mathbb{P}(B) \neq 0$. Show that :

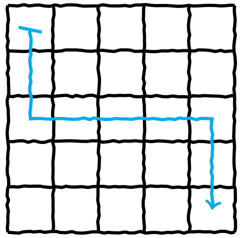
$$\mathbb{P}(A) = \mathbb{P}_B(A)\mathbb{P}(B) + \mathbb{P}_{\overline{B}}(A)\mathbb{P}(\overline{B}).$$

20 In a rock-paper-scissors game, if both player choose randomly (uniformly), what is the probability of a tie ? What about rock-paper-scissors-lizard-Spock[2] ?

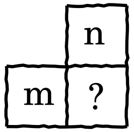
21 Every friday morning, a teacher struggles with calculus at the board. Given that his students make a lot of noise, he has a 50%-chance of making a mistake. But when they are quiet, he only has a 20%-chance of making a mistake. The probability for the students to blabber is 40%. Last friday, the teacher did a computational mistake. What is the probability that his students were noisy ?

22 How many paths in this grid can you find that satisfy all of these three conditions ?

1. start in the upper left corner ;
2. end in the lower right corner ;
3. only go right or down.



Hint : Write in each square how many different paths lead there and complete this partial diagram :



23 A fair die has the number 1 printed on three of his faces, the number 2 printed on two others and the number 3 on the last one. Denote by X the result of a roll.

a. What is the distribution of X ?

b. What is the expectation of X ?

c. What is the variance of X ?

24 Two fair dice are rolled. Denote by X the sum of the outcomes.

a. What is the probability distribution of X ?

b. What is the cumulative distribution function of X ?

1. Two Face was once Harvey Dent before becoming a well-known supervillain in Batman.
2. rock-paper-scissors-lizard-Spock is the famous lizard-Spock expansion of the rock-paper-scissors game. All Hail Sam Kass !

c. Compute $\mathbb{E}(X)$ and $\mathbb{V}\text{ar}(X)$

25 Let X be the random variable counting the number of boys in a family of 5 children. Assuming equiprobability between boys and girls :

a. What is the distribution of X ?

b. Compute the probability that there is at least one boy.

c. Compute the probability that there is at least one boy and at least one girl.

26 An urn contains 12 marbles : 6 are green, 4 are red and 2 are blue. Give the probability distribution corresponding to the following experiments :

a. You draw randomly with replacement 8 marbles and denote by X the total number of red marbles.

b. You draw randomly without replacement 8 marbles and denote by Y the total number of red marbles.

c. You draw randomly with replacement 1 marble at a time until you get a red one and denote by Z the total number of trials.

27 An urn contains n marbles numbered $1,2,\dots,n$. One draws simultaneously 2 marbles and denote by X the maximum of the numbers.

a. Compute the cumulative distribution function of X.

b. Compute the probability mass function of X.

28 Let X be a random variable such that $X(\Omega) = [\![1,20]\!]$ and :

$$\forall k \in X(\Omega), \quad \mathbb{P}(X = k) = \frac{k}{210}.$$

Compute the probability $\mathbb{P}(X > 17)$.

29 Let X be a random variable satisfying :

x_i	-1	0	1	2
$\mathbb{P}(X = x_i)$	.2	.1	.3	.4

Compute the expectation and variance of X.

30 Let $X \sim \mathscr{B}(n,p)$ and $Y \sim \mathscr{B}(m,p)$ be two independant random variables. What is the probability distribution of $Z = X + Y$?

31 An experiment may succeed with probability 0.08. It is performed until we have a success, and X denotes the number of trials until the first success.

a. What is the distribution of X ? Give its expectation and its variance.

b. Compute $P(X > 3)$ and $P_{(X>3)}(X > 6)$.

c. Find the smallest integer n such that $P(X \geqslant n) \leqslant 0.05$.

d. We now decide to perform 50 times the experiment. The random variable Y counts the number of successes. What is the distribution of Y ? Compute $P(Y \geq 5)$.

32 You roll a fair die and denote by X the result. You then flip a coin X times and denote by Y the number of heads.

a. Find the probability mass function of X.

b. Find the probability mass function of Y (hint : given $X = n$, what is the probability mass function of Y ?).

33 A bag cointains 26 chips labeled a,b,c,...,z. One draws simultaneously 5 chips from the bag and counts the number X of vowels (a,e,i,o,u and y). What is the probability distribution of X ?

34 Let $n \in \mathbb{N}^\star$ and suppose you toss a fair coin $2n$ times.

a. What is the probability p_n of getting as many times heads and tails (exactly n heads and n tails)?

b. Compute the probability of getting strictly more heads than tails.

c. Compute the probability of getting an odd number of heads.

35 The number X of tankers entering the seaport of Marseille on a given day is Poisson-distributed with parameter 2. The port can handle 2 boats a day. When 3 boats or more present themselves on one day, the overflow is sent to another port. What is the probability that at least one tanker is sent away on a given day?

36 A mineral water factory has a great stock of bottles, 7.5% of which contain more than 6.5mg of limestone per one liter. One draws randomly 40 bottles in the stock which is big enough so that one can consider it to be a draw *with replacement*. We denote by X the total number of bottles with a big limestone concentration (more than 6.5mg/l).

a. What is the probability distribution of X ?

b. Suppose this distribution were to be approximated using a Poisson distribution, what would its parameter λ be?

c. Let $Y \sim \mathscr{P}(\lambda)$. Compute $\mathbb{P}(Y \leqslant 4)$ and compare it to $\mathbb{P}(X \leqslant 4)$.

37 The number N of children in a family of super-heroes is supposed to be Poisson-distributed with parameter m. When a child is born, he has a probability p of having a super-power, independantly from N. Let X be the number of children with super-powers and Y the number of *normal* children.

a. What relation links N, X and Y ?

b. Suppose $m = 2$ and $p = .4$. What is the probability for plastic man and invisible woman to have exactly 3 kids with super-powers and 2 regular ones.

c. Let $n \in \mathbb{N}$ and $k \in [\![0, n]\!]$. Compute $\mathbb{P}(X = k | N = n)$.

d. Using the formula

$$e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$$

and the conditional law of total probabilities, compute the distribution of X.

e. What is the probability that the fantastic 4 will become the fantastic 6 ?

38 Let $X \sim \mathscr{G}(p)$ and $(n, m) \in \mathbb{N}^2$. Proove that :

$$\mathbb{P}(X > n + m | X > m) = \mathbb{P}(X > n).$$

39 A lab rat can choose between 5 doors. If he picks the good one, he gets to eat cheese. If he doesn't, he gets tased. The rat chooses a door randomly. The experiment stops when the rat gets to the cheese.

1. Suppose that the electrical shocks make him forget what door he picked. What is the probability that he will get to the cheese on his 6th attempt ?
2. Suppose he remembers which doors not to open after being tased. Find the probability distribution of the total number X of doors he will open to get to the cheese. Compute $\mathbb{E}(X)$.
3. Suppose he only remembers what's behind the last door he opened. Find the probability distribution of the total number X of doors he will open to get to the cheese.

40 Let $a \in \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ satisfy :

$$f(x) = \begin{cases} ax(1-x) & \text{if } x \in [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

1. Plot f.
2. Find a such that f is a probability density function.
3. Let X be a random variable with probability density function f.
   (a) Compute $\mathbb{P}(X \leqslant .5)$.
   (b) Compute $\mathbb{E}(X)$ and $\mathbb{V}\text{ar}(X)$.

41 Let $f : \mathbb{R} \to \mathbb{R}$ be the map defined by :

$$f(x) = \begin{cases} xe^{-x^2/2} & \text{if } x \geqslant 0, \\ 0 & \text{otherwise.} \end{cases}$$

1. Show that f is a probability density function.
2. Suppose that X has density f.
   (a) Compute the cumulative distribution function of X.
   (b) Compute the midian of X, that is, the real number m such that $\mathbb{P}(X < m) = \mathbb{P}(X > m)$.

(c) Using results on the normal distribution, prove that $\mathbb{E}(X)$ exists and compute it.

42 Let X be a random variable with cumulative distribution function F given by :

$$F(x) = \begin{cases} 1 - \dfrac{1}{(x+1)^2} & \text{if } x \geqslant 0, \\ 0 & \text{otherwise.} \end{cases}$$

1. Show that X is a continuous random variable and find its probability density function.
2. Compute $\mathbb{P}(X \geqslant 2)$ and $\mathbb{P}(-3 < X \leqslant 4)$.
3. Compute $\mathbb{E}(X)$ and $\mathbb{V}\mathrm{ar}(X)$ if they exist.
   Hint : $\frac{x}{(1+x)^3} = \frac{1+x}{(1+x)^3} + \frac{-1}{(1+x)^3}$.

43 The lifespan X of the radioactive iodine-131 isotope is exponentially distributed with parameter λ. We know that its half-life is approximately 8 days : $\mathbb{P}(X > 8) = \mathbb{P}(X < 8)$.

1. Compute λ. What is the average lifespan of the radioiodine ?
2. Given that a nuclide is still not disintegrated after 4 days, what is the probability that its lifespan will be greater that 20 ?
3. Let $n \geqslant 2$. One puts n nuclides of radioiodine in a closed chamber. Assume that the disintegrations are mutually independent and that the lifespans are still exponentially distributed with the same parameter. Denote by N_t the total number of non-desintegrated nuclides at time t (where t is the number of days).
   (a) Show that the distribution of N_t is a binomial one.
   (b) Compute $\mathbb{E}(N_t)$.
   (c) Compute $\mathbb{V}\mathrm{ar}(N_t)$.
   (d) Compute $\mathbb{P}(N_t \geqslant 2)$.

44 Let $U \sim \mathcal{U}([0,1])$ and let $V = 1 - U$.

1. Compute the cumulative distribution function of V.
2. Is V continuous ? If so, compute its probability density function.
3. Are U and V independent ?

45 Let :

$$f(t) = \begin{cases} C(1-t^2) & \text{if } t \in [-1,1], \\ 0 & \text{otherwise.} \end{cases}$$

1. Can f be a probability density function ?
2. Let X be a random variable with density f. Compute its cumulative distribution function and plot it.
3. Compute $\mathbb{P}(X \leqslant 1)$, $\mathbb{P}(0 \leqslant X \leqslant 1)$ and $\mathbb{P}(X \geqslant \frac{1}{2})$.

46 Let :

$$f(t) = \begin{cases} 10/t^2 & \text{if } t > 10, \\ 0 & \text{otherwise.} \end{cases}$$

1. Plot f.
2. Show that f is a probability density function.
3. Let X be a random variable with probability density function f. Compute $\mathbb{P}(X > 10)$ and $\mathbb{P}(X > 20)$.
4. Compute the corresponding cumulative distribution function.
5. Suppose X has density f. Does $\mathbb{E}(X)$ exist ?

47 Let $X \sim \mathcal{U}([0,1])$ and $Y = X^2$.

1. Compute the cumulative distribution of Y.
2. Is Y a continuous random variable ? If so, compute its density.
3. Compute $\mathbb{P}(Y \geq 0)$ and $\mathbb{P}(Y \leq \frac{1}{4})$.

48 Let X and Y be two independent random variables with distribution $\mathcal{U}([0,1])$.

1. Compute the c.d.f. of $Z = \max(X,Y)$.
2. If Z is continuous, compute its probability density function.
3. Generalize the result to $Z = \max(X_1,\ldots,X_n)$ where the X_i's are independent random variables with distribution $\mathcal{U}([0,1])$.

49 Let $f : \mathbb{R} \to \mathbb{R}$ satisfy :

$$f(x) = \begin{cases} |x| & \text{if } -1 \leqslant x \leqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

1. Plot f.
2. Show that f is a probability density function
3. Compute the corresponding cumulative distribution function.
4. Let X be a continuous random variable with probability density function f. and let :

$$Y = \begin{cases} 1 & \text{if } X > 0, \\ 0 & \text{otherwise.} \end{cases}$$

What is the probability distribution of Y ?

50 Let $X \sim \mathcal{E}(\lambda)$ and $(h,t) \in \mathbb{R}_+^2$. Show that :

$$\mathbb{P}(X \geqslant t+h | X \geqslant t) = \mathbb{P}(X \geqslant h).$$

51 Let $X \sim \mathcal{N}(\mu, \sigma^2)$.

1. Compute $\mathbb{P}(|X-\mu| \leqslant \sigma)$.
2. Compute $\mathbb{E}(X^2)$.

52 In a pastry factory, one weights cookies to check their quality. We suppose that the weight of a cookie (in grams) is normaly distributed with parameters (15,.4).

1. What is the probability that a randomly chosen cookie weights more than 15g ?
2. A cookie is considered of good quality if its weight is between 14 and 16 grams.
   (a) In his ad, the seller says that 95% of his cookies are of good quality. Can we trust him?
   (b) What number would you give?

53 A factory produces aluminum bars. We assume that the length X of an aluminum bar satisfies $X \sim \mathcal{N}(m, \sigma^2)$ where $m = 54$ and $\sigma = 0.2$. A bar is defective when its length x is not in the interval $[53.6, 54.3]$.

1. What is the probability that a bar is defective?
2. You pick randomly 20 bars in the stock. What is the probability that at least three of them are defective?

54 Let X be a continuous random variable with probability density function f_X and with cumulative distribution function F_X. What are the cdfs and pdfs of the following random variables?

1. $Y_1 = aX + b$ where $a > 0$.
2. $Y_2 = |X|$.
3. $Y_3 = X^2$.
4. $Y_4 = \exp(X)$.

55 Let $U \sim \mathcal{U}([0,1])$.

1. Compute $\mathbb{P}(U = 0)$.
2. Let $V = -\ln(U)$. Compute its cumulative distribution fonction. What is the probability distribution of V ?
3. Same question with $V = -\frac{1}{\lambda}\log(U)$ where λ is positive.

56 An urn contains four chips labeled 1 through 4. One draws two chips and denote by X and Y the first and second number respectively.

1. Write the joint probability table of (X, Y).
2. Compute $\mathbb{C}\text{ov}(X, Y)$ and $\mathbb{V}\text{ar}(X)$.
3. Are X and Y independent?

57 The joint probability function of two discrete random variables X and Y is given by :

$$f(x,y) = \begin{cases} c(2x+y) & \text{if } (x,y) \in [\![0,2]\!] \times [\![0,3]\!], \\ 0 & \text{otherwise.} \end{cases}$$

1. Write the joint probability table of (X, Y).
2. Find the value of c.
3. Find $\mathbb{P}(X = 2, Y = 1)$

4. Find $\mathbb{P}(X \geq 1, Y \leq 2)$.
5. Find the marginal probability function of X and Y.
6. Are the variables independent ?

58 Let $X_1, X_2, \ldots, X_n$ be n independent random variables having the same distribution, with expectation μ and variance σ^2. Compute the expectations and variances of the following variables :

1. $Y_1 = nX_1$
2. $Y_2 = \sum_{i=1}^{n} X_i$.
3. $Y_3 = \alpha X_1 + \beta X_2$, with $(\alpha, \beta) \in \mathbb{R}^2$.
4. $Y_4 = \frac{1}{n} \sum_{i=1}^{n} X_i$.

59 Let $X \sim \mathscr{U}([-1,1])$ and $Y = X^2$.

1. Compute $\mathbb{C}\text{ov}(X,Y)$.
2. Let $A = [0,1/2]$ and $B = [3/4,1]$. Compute $\mathbb{P}(X \in A, Y \in B)$ and $\mathbb{P}(X \in A)\mathbb{P}(Y \in B)$.
3. Are X and Y independent?

60 Let X and Y be two independent Bernoulli variables with the same parameter p. Let also $U = X + Y$ and $V = X - Y$.

1. Compute the joint probability distribution of (U,V).
2. Compute the covariance of U and V.
3. Are U and V independent?

61 Let X be a random variable with distribution $\mathscr{U}(\{-a,-b,a,b\})$ where $a \neq b$ are two real numbers. Let also $Y = X^2$.

1. Write the joint probability table of (X,Y).
2. Show that $\mathbb{C}\text{ov}(X,Y) = 0$.
3. Are X and Y independent?

62 Suppose you roll a die and win 1€ if it lands on a 4, 4€ if it lands on a 5 and lose 1€ otherwise. Denote by X the result of the roll and by Y your winnings.

1. Write the joint probabilities table of (X,Y).
2. Compute $\mathbb{C}\text{ov}(X,Y)$.
3. Are X and Y independent?

63 Let $X \sim \mathscr{G}(p_1)$ and $Y \sim \mathscr{G}(p_2)$ be two independent random variables.

1. Compute the cdf of X.
2. Let $U = \min(X,Y)$.
   (a) Compute $\mathbb{P}(U > k)$ for $k \in \mathbb{N}$.
   (b) Compute $\mathbb{P}(U = k)$ for $k \in \mathbb{N}^\star$.
   (c) Show that $U \sim \mathscr{G}(1-(1-p_1)(1-p_2))$.
   (d) Generalize this result to n mutually independent geometric variables.

64 Let $X \sim \mathscr{E}(\lambda)$ and $Y \sim \mathscr{E}(\mu)$ be two independent random variables.

1. Compute the cdfs of X and Y.
2. Let $U = \min(X,Y)$.
   (a) Compute $\mathbb{P}(U > t)$ for $t > 0$.
   (b) What is the probability distribution of U?
   (c) Generalize this result to n mutually independent exponential variables.

65 Let X and Y be two independent random variables with respective cdfs F_X and F_Y, and let $U = \max(X,Y)$.

1. Compute the cdf of U.
2. Show that if X and Y are continuous then U is also continuous and give its pdf.

66 Let $X \sim \mathscr{U}([0,1])$ and let Y be a random variable independent from X. Suppose that :

$$\mathbb{P}(Y = 1) = \mathbb{P}(Y = -1) = \frac{1}{2}.$$

Let $Z = XY$.

1. Compute $\mathbb{P}(Z \leqslant t, Y = 1)$ for all $t \in Z(\Omega)$.
2. Compute $\mathbb{P}(Z \leqslant t, Y = -1)$ for all $t \in Z(\Omega)$.
3. Using the law of total probabilities, compute $\mathbb{P}(Z \leqslant t)$ for all $t \in Z(\Omega)$.
4. Show that Y is continuous and compute its pdf.
5. Suppose now that :

$$\mathbb{P}(Y = 1) = 1 - \mathbb{P}(Y = -1) = p,$$

where $p \in (0,1)$ is fixed. Compute the pdf of Y.

67 Let $X \sim \mathcal{U}([-1,1])$ and :

$$Y = \begin{cases} 1 & \text{if } X \geqslant 0, \\ -1 & \text{otherwise.} \end{cases}$$

Let $Z = XY$. Compute the cdf of Z and recognize its probability distribution.

68 Let X and Y be two independent Bernoulli variables with parameters p and q respectively.

1. What is the probability distribution of $Z_1 = XY$?
2. What is the probability distribution of $Z_2 = \max(X,Y)$?
3. What is the probability distribution of $Z_3 = \min(X,Y)$?

69 Let X and Y be two independent random variables satisfying :

$$\mathbb{P}(X = 1) = \mathbb{P}(Y = 1) = 1 - \mathbb{P}(X = -1) = 1 - \mathbb{P}(Y = -1) = p,$$

where $p \in [0,1]$.

1. What is the probability distribution of $Z_1 = XY$?
2. What is the probability distribution of $Z_2 = \max(X,Y)$?
3. What is the probability distribution of $Z_3 = \min(X,Y)$?

70 Let $X \sim \mathcal{U}([-1,1])$ and let $Y = |X|$.

1. Compute the cdf of X.
2. Compute the cdf of Y.
3. What is the probability distribution of Y given $X \geqslant 0$?

71 Let X and Y be two independent random variables with the same distribution $\mathcal{U}([0,1])$. Compute the probability density function of $Z = X + Y$.

72 Let X and Y be two independent random variables with respective distributions $\mathcal{E}(\lambda)$ and $\mathcal{E}(\mu)$. Compute the probability density function of $Z = X + Y$.

73 During an experiment, an event has a probability $p = 0.2$ to come true.

1. We repeat the experiment n times and suppose that the results are mutually independent. If X represents the total number of successes, what is the distribution of X, its expected value and variance ?
2. We suppose that $n = 100$. Compute $\mathbb{P}(15 \leqslant X \leqslant 25)$ by :
   (a) using X's distribution.
   (b) approximating X's distribution with a normal one.

74 Let $(X_i)_i$ be a sequence of independent and identically distibuted random variables with $X_i \sim X$ and let $A \in X(\Omega)$ be an event.

1. Let

$$Y_i = \begin{cases} 1 & \text{if } X_i \in A \\ 0 & \text{othewise} \end{cases}$$

   Compute $\mathbb{E}(Y_i)$ and $\mathbb{V}\text{ar}(Y_i)$.
2. What is the meaning of $\frac{1}{n}\sum_{i=1}^{n} Y_i$?
3. Conclude that the frequency of occurences of A converges in probability.

75 Let Y denotes the sum of 20 fair dice rolls. Give an approximation of $\mathbb{P}(60 \leq Y \leq 75)$.

76 Let (X_n) be a sequence of independent Poisson random variables with parameter $\lambda = 1$.

1. What is the limit of $Y_n = \dfrac{X_1 + \cdots + X_n - n}{\sqrt{n}}$ in distribution?
2. Deduce $\lim\limits_{n \to +\infty} \sum\limits_{k=0}^{n} \dfrac{n^k e^{-n}}{k!}$.
3. Suppose $Y \sim \mathscr{P}(20)$.
   (a) Compute $\mathbb{P}(16 \le Y \le 20)$.
   (b) Give a normal approximation of $\mathbb{P}(16 \le Y \le 20)$.

77 Let $A \subset [0,1] \times [0,1]$. If X and Y are two independent $\mathscr{U}([0,1])$ random variables, then, one can show that :

$$\mathbb{P}((X,Y) \in A) = \mathscr{A}(A),$$

where $\mathscr{A}(A)$ is the area of A. Let $f : [0,1] \to [0,1]$.

1. Let X and Y be two independent $\mathscr{U}([0,1])$ random variables. Compute $\mathbb{P}(Y \leqslant f(X))$ (make a drawing).
2. Let $(X_n)_n$ and $(Y_n)_n$ be sequences of mutually independent random variables with the same distribution $\mathscr{U}([0,1])$. For all $n \in \mathbb{N}^\star$, let :

$$Z_n = \begin{cases} 1 & \text{if } Y_n \leqslant f(X_n), \\ 0 & \text{otherwise.} \end{cases}$$

Let also :

$$S_n = Z_1 + Z_2 + \ldots + Z_n.$$

   (a) What is the probability distribution of S_n?
   (b) What is the limiting distribution of :

$$S_n^\star = \frac{S_n - \mathbb{E}(S_n)}{\sqrt{\mathbb{V}\mathrm{ar}(S_n)}}.$$

   From now on, we will assume that n is large enough to approximate the distribution of $S_n^\star$ with the above limiting distribution.

   (c) What approximation can we use for the distribution of S_n/n?
   (d) Approximate $\mathbb{P}(|S_n^\star| \leqslant 1.96)$.
   (e) Show that :

$$\mathbb{P}\left(\left|\frac{S_n}{n} - \int_0^1 f(t)\mathrm{d}t\right| \leqslant \frac{1}{\sqrt{n}}\right) \geqslant 0.95.$$

78 Let $(X_n)_n$ be a sequence of i.i.d. random variables with distribution $\mathscr{N}(0,1)$. We denote by :

$$T_n = \frac{1}{n}\sum_{k=1}^{n} X_k^2.$$

Show that $(T_n)_n$ converges in probability.

79 Someone mixed 5,000 NPN transistors with 10,000 PNP transistors. You pick randomly 150 transistors. What is the probability that you will get between 45 and 60 NPN transistors?

80 Let $(x_1, \ldots, x_{100})$ be an observed sample of some random variable X. Suppose that :

$$\sum_{k=1}^{100} x_k = 2000 \quad \text{and} \quad \sum_{k=1}^{100} x_k^2 = 41062.$$

Give an estimation of $\mathbb{E}(X)$ and $\mathbb{V}\mathrm{ar}(X)$.

81 Coffee beans bags are weighted at the end of a production line. The results are shown in the table below.

mass (g)	247	248	249	250	251	252	253	254
number of bags	2	6	8	13	11	5	3	2

1. Give an estimation of the mean weight and variance of a bag.
2. Suppose that the weight of a bag is normally distributed. Using the approximations from question 1, estimate the proportion of bags that weight more than 250g.

82 We try to estimate the proportion of people p who own an eye-

Phone. A radomly selected sample $(X_1,\ldots,X_n)$ is taken in a huge population. Each random variable X_i is defined by :

$$X_i = \begin{cases} 1 & \text{if person } i \text{ owns an eyePhone,} \\ 0 & \text{otherwise.} \end{cases}$$

1. Give an estimator $T(X_1,\ldots,X_n)$ of p, what is its bias ? Is it consistent?
2. Suppose that $(X_1,\ldots,X_{n_1})$ and $(X'_1,\ldots,X'_{n_2})$ are two random samples whose sizes are n_1 and n_2 ($n_1 < n_2$). We denote by f_1 and f_2 the proportions of people who own an eyePhone player in the first and second sample. Let $F = \alpha F_1 + \beta F_2$ with $\alpha, \beta > 0$ be an estimator of p.
   (a) Find α and β so that F is unbiased.
   (b) Compute $V(F)$.
   (c) What values of α and β minimize the variance of F and leave it unbiased?
   (d) Compute F with the following values :

$$n_1 = 500,\ n_2 = 1000,\ f_1 = 0.7 \text{ and } f_2 = 0.64.$$

83 Let $X \sim \mathscr{U}([0,\theta])$ and let $(X_1,\ldots,X_n)$ be a random sample of X. Let also

$$T_n(X_1,\ldots,X_n) = \max(X_1,\ldots,X_n)$$

and

$$S_n(X_1,\ldots,X_n) = \frac{2}{n}\sum_{i=1}^{n} X_i$$

be two estimators of θ.

1. Study the properties of S_n (consistency and biasedness).
2. Compute the cdf of T_n. Deduce a pdf of T_n and study the consistency and biasedness of T_n.

84 Let $(X_1,\ldots,X_n)$ be a sample of $X \sim \mathscr{P}(\lambda)$.

1. Show that
$$\overline{X}_n = \frac{X_1 + \ldots + X_n}{n}$$
is a consistent and unbiased estimator of λ.
2. Show that
$$T_n = \frac{2}{n(n+1)}\sum_{k=1}^{n} kX_k$$
is also a consistent and unbiased estimator of λ.
3. Which of the two estimators is the "best"?

85 A sample of size 20 has been taken in a mass production. We assume that the population is normal $\mathscr{N}(\mu,\sigma^2)$, with μ and σ^2 unknown. The observations are given below :

5.5, 5.8, 6.1, 6.5, 5.8, 5.8, 5.5, 6.1, 5.7, 5.4,
5.5, 5.9, 6.2, 6.1, 5.8, 6.1, 5.9, 6.1, 6.2, 6

1. Give a point estimation of μ and σ.
2. Give a confidence interval for σ^2 with $\alpha = 0.01$. Is there strong evidence to say that the standard deviation of the population is smaller than 0.5?
3. Give a 99%-confidence interval for μ.

86 A sample of 500 teenagers has been selected and 210 of them were overweight. Let p be the proportion of overweight teenagers. Give two confidence intervals for p at respective levels 95% and 99%.

87 The article "Gas Cooking, Kitchen Ventilation and Exposure to Combustion Products" (*Indoor Air*, 2006) reported that for a sample of 50 kitchens with gas cooking appliances monitored during a one-week period, the sample mean of CO_2 level (ppm[3]) was 654.16, and the sample standard deviation was 164.43.

3. PPM=parts-per-million. This notation is used in engineering to measure low proportion, 1 ppm=0,0001%. A CO_2 concentration between 300 and 2,500 ppm is used as an indicator of indoor air quality

1. Calculate and interpret a 95% confidence inteval for the true average CO_2 level in the population of all home from which the sample was selected.
2. What sample size would be necessary to obtain an interval width of 50 ppm for a confidence level of 95% ?

88 Before election day, ones makes an opinion poll :

1. In a randomly selected sample of 1000 people, 500 claim to be up to vote for A, 250 want to vote for B and 50 for C. Give 95%-confidence intervals for the proportion of people who want to vote for A,B,C.
2. A fourth candidate D gets 17% of vote intention. How many people should be questionned to obtain a 95%-confidence interval with width less than 0.02 (i.e. precision of ±1%).

89 The SAT scores from a random sample of 91 high school seniors were analyzed and found to have a mean of 545 and a standard deviation of 75. Find a 90% confidence interval.

90 An IQ test was administered to a random sample of 101 sixth graders. The sample mean was 104.2 with a standard deviation of 12. Find a 99% confidence interval for the mean.

91 The average height of a random sample of 41 adult females was 66 inches with a standard deviation of 1.9 inches. Find a 95% confidence interval for the height.

92 A sample of 100 people has been drawn from the population of a town. There are 44 men and 56 women in the sample. Test the hypothesis that the ratio of men to women is equal to 50% with a risk level of your choice.

93 A survey claims that 9 out of 10 doctors recommed aspirin for their patients with headaches. To test this claim, a random sample of 100 doctors is obtained. Of these 100 doctors, 82 indicate that they recommand aspirin. Is this claim accurate? Use $\alpha = 0.05$

94 The CEO of a large electric utility claims that 80 percent of his 1,000,000 customers are very satisfied with the service they receive. To test this claim, the local newspaper surveyed 100 customers. Among the sampled customers, 73 percent say they are very satisfied. Based on these findings :

1. Give a point estimate of p, the proportion of "very satisfied customers"
2. Build a 99%-confidence interval for p
3. Can we reject the CEO's hypothesis? Use a risk level of 0.05.

95 A factory produces regular dice and wants to check that they are fair. In order to do so, an employee takes one die at random and rolls it 40 times. Here are the results :

1	2	3	4	5	6
5	4	3	7	9	12

Test whether the die is fair or not with a risk level of 1% and then 5%.

Some of these exercises were take from : `www.statlect.com/`. You can find more exercises (with solutions) on this website. Great video courses and exercises are also available at `www.khanacademy.org`, where a few exercises were also borrowed.