1 Draw 1000 realizations of a $\mathscr{U}([0,1])$ random variables and plot the corresponding histogram using the `histplot` function.

2 Repeat the previous exercise using the `grand` function for the different continuous distributions from the course (exponential and normal) and plot the probability density function of the corresponding probability distributions on the same graphs.

3 Let $U \sim \mathscr{U}([0,1])$ and let $\lambda > 0$. One can show that :

$$-\frac{1}{\lambda}\ln(U) \sim \mathscr{E}(\lambda).$$

To test this claim, draw 10,000 numbers with the same $\mathscr{U}([0,1])$ distribution and transform them using the above function. Plot a histogram and compare it to the probability density function of the $\mathscr{E}(\lambda)$ distribution.

4 Let U_1 and U_2 be two independent $\mathscr{U}([0,1])$ variables. One can show that

$$\sqrt{-2\ln(U_1)}\cos(2\pi U_2) \sim \mathscr{N}(0,1).$$

Test this claim by adapting the technique seen in exercise 3.

5 Let $X \sim \mathscr{E}(\lambda)$ where $\lambda > 0$. Let :

$$Y = \lfloor X \rfloor + 1.$$

One can show that Y is a geometric variable with parameter $(1-\mathrm{e}^{-\lambda})$. Test this claim by adapting the technique seen in exercise 3.
Hint : instead of using the `histplot` function, use the `plot2d3` one and/or something like `plot2d(x,y,-1)`. Plus, you will probably need the `tabul` function.

6 Let U_1 and U_2 be two independent $\mathscr{U}([0,1])$ random variables and let $S = U_1 + U_2$. One can show that S is a continuous random variable with a probability density function shaped like a triangle. Test this claim by adapting the technique seen in exercise 3.

7 Let $U \sim \mathscr{U}([0,1])$ and $A \subset [0,1]$ then $\mathbb{P}(U \in A)$ is equal to the area of the set A as :

$$\mathbb{P}(U \in A) = \int_A \mathbb{1}_{[0,1]}(t)\,\mathrm{d}t.$$

1. Generate a "large" number of realizations of the $\mathscr{U}([0,1])$ distribution and compute the frequency of numbers in a subset A of $[0,1]$ of your choice.
2. Generate a "large" number of realizations of the $\mathscr{U}([0,1]\times[0,1])$ and compute the frequency of numbers in a subset A of $[0,1]\times[0,1]$ of your choice.
   Hint : if U_1 and U_2 are independent with the same $\mathscr{U}([0,1])$ distribution, then $U = (U_1, U_2)$ is uniformly distributed on $[0,1]\times[0,1]$.

8 Repeat exercise 6 with a sum S of 15 $\mathscr{U}([0,1])$ random variables. Does the shape of the histogram look like a probability density function from the course? Plot it against the histogram.
Hint : $\mathbb{V}\mathrm{ar}(V) = 15/12$ and $\mathbb{E}(V) = 7.5$.

9 Let X and Y be two independent exponential variables with parameters λ and μ. One can show that $Z = \min(X,Y)$ is an exponential random variable with parameter $\lambda + \mu$. Test this claim by adapting the technique seen in exercise 3.

10 The χ^2 (chi-squared) distribution is defined as a sum of squares of independent $\mathscr{N}(0,1)$ variables : if $X_1, \dots, X_k$ are independent and $\mathscr{N}(0,1)$-distributed, then :

$$Z = \sum_{i=1}^{k} X_i^2 \sim \chi_k^2.$$

Answer the following questions for $k = 1,2,3,4$.

1. Draw a "large number" of vectors $(x_1, \dots, x_k)$ of $\mathscr{N}(0,1)$ realizations and plot a histogram of the corresponding "large number" of χ_k^2 realizations (you can use the `grand` function).
2. A pdf of the χ_k^2 distribution is given by :

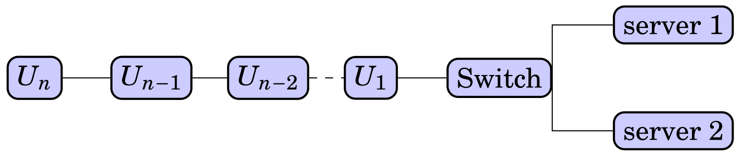
$$f(x) = \frac{1}{2^{\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)} x^{\frac{k}{2}-1} e^{-\frac{x}{2}} \mathbb{1}_{\{x>0\}}.$$

   Using the `gamma` function to compute $\Gamma(k/2)$, plot this function against the histogram.

3. Check that $\mathbb{E}(Z) = k$ by computing the mean value of a "big" sample.

11 If X and Y are two independent Poisson-distributed variables with parameters λ and μ, then $X + Y \sim \mathscr{P}(\lambda + \mu)$. Test this claim using the `grand` function and the technique seen in exercise 3. As the variables are discrete, you should use `plot2d3` instead of `histplot`.

12 Suppose two servers have to deal with an (infinite) sequence of tasks. Task n needs a time U_n to be dealt with. We suppose that the variables U_n are independent and have a $\mathscr{U}([0,1])$ distribution. We will try two strategies in order to allocate the tasks evenly to the servers.



Denote by :

$$a_n = \begin{cases} 1 & \text{if task } n \text{ is assigned to server 1,} \\ -1 & \text{if task } n \text{ is assigned to server 2.} \end{cases}$$

Then, the load difference between the two servers for the N first tasks is :

$$D_N = \sum_{n=1}^{N} a_n U_n.$$

1. When a task comes in, it can be assigned randomly to a server (server 1 with probability 0.5 and server 2 with probability 0.5).
   (a) Create a `scilab` function that simulates this process and computes the load difference for $N = 100$.
   (b) Repeat the experiment a "large number" of times and plot the corresponding histogram.
   (c) One can show that in this set-up, the distribution of D_N is approximately a $\mathcal{N}(0, N/3)$: test this claim by plotting a pdf of this distribution against the histogram.
2. Instead of assigning the tasks randomly, a batter strategy is to assign them depending on the current load difference :
   - choose a_1 randomly
   - if $a_1, \ldots, a_{n-1}$ are chosen, chose a_n to be -1 or 1 depending on whether D_{n-1} is positive or negative.

   (a) Create a `scilab` function that simulates this process and computes the load difference for $N = 100$.
   (b) Repeat the experiment a "large number" of times and plot the corresponding histogram.
   (c) Plot the following function against the histogram :

$$\begin{array}{rccc} f & : & \mathbb{R} & \rightarrow & \mathbb{R} \\ & & x & \mapsto & (1 - |x|)\mathbb{1}_{[-1,1]}(x) \end{array}$$